AIAA JOURNAL
Vol. 34, No. 7, July 1996

Bimodal Bound of System Reliability for Random Composite Structures

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A procedure of system reliability analysis using the stochastic finite element method for composite structures is presented. Composite structure is considered as a random structure. The material properties, fiber angles, laminate thickness, and curvatures are modeled as random parameters. The curvature means initial curvature as a result of manufacturing error for the case of a flat plate. The random thickness and fiber angles affect the strain-displacement relation and material directions. In the stochastic finite element formulation, these randomnesses are considered to obtain accurate statistical responses. In reliability analysis, composite laminates are assumed to be multicomponent series system, and the correlations between possible failure modes of different components are considered to obtain bimodal bounds of failure probability. The method is verified through comparison of the results with those obtained by the Monte Carlo simulation for two example problems. The developed method is efficient and accurate enough to be applied to engineering problems and can assess the risk of structures quantitatively and logically.

Nomenclature

$\boldsymbol{B}$ = matrix of strain-displacement relation
b_i = vector of discretized random variables for random field
$b(\boldsymbol{x})$ = random field
$\boldsymbol{C}$ = material stiffness matrix
$\mathrm{Cov}(\cdot,\cdot)$ = covariance of random quantities
E_i = event of ith single mode failure in a laminate
$E[\cdot]$ = mean value of random quantities
$\boldsymbol{F}$ = nodal force vector in finite element equations
g_i = limit-state function of ith single failure mode
$\boldsymbol{K}$ = global stiffness matrix in finite element equations
m = number of different failure modes in a laminate
$P(\cdot)$ = probability of any event
P_f = probability of system failure
P_{f_i} = probability of ith single mode failure
$\boldsymbol{Q}$ = reduced stiffness matrix of lamina
$\boldsymbol{Q}'_k$ = transformed reduced stiffness matrix of kth lamina
$R(\cdot,\cdot)$ = autocorrelation coefficient function of random field
$SD[\cdot]$ = standard deviation of random quantities
$\boldsymbol{T}$ = transformation matrix from material coordinates of lamina to global coordinates of laminate
$\boldsymbol{u}$ = nodal displacement vector in finite element equations
$\mathrm{Var}(\cdot)$ = variance of random variable
X_i, Y_i = random variables of stress and strength corresponding to ith failure mode
$\boldsymbol{x}$ = vector of spatial coordinates
β_i = reliability index for ith single failure mode
ε, σ = strain and stress vector in global coordinates of laminate, respectively
ε', σ' = strain and stress vector in material coordinates of lamina
λ_x, λ_y = correlation decay factors in x and y directions, respectively
μ_{X_i}, μ_{Y_i} = mean values of random variables X_i and Y_i, respectively
ρ_{ij} = correlation coefficient between ith and jth failure modes
$\sigma_{X_i}, \sigma_{Y_i}$ = standard deviations of random variables X_i and Y_i, respectively
Φ = cumulative distribution function of standard normal distribution

Introduction

COMPOSITE structures have inherent large uncertainties and variabilities in their structural properties. Therefore, the deterministic structural analysis and design may not be sufficient to assure safety and efficiency of composite structures. Furthermore, determining an appropriate value of safety factor is very difficult because there are often insufficient accumulated data. In this regard, the effective methods of probabilistic analysis and reliability estimation considering the uncertainties of a structure itself are needed. Composite structures should be modeled as random structures with uncertain parameters in this approach. The reliability analysis for random structures can be made effectively based on the second moment approach, in which the first two statistical moments of structural responses are used and they can be most efficiently evaluated using the stochastic finite element method (SFEM).

In the SFEM based on the second moment method, random parameters of a structure are modeled as stochastic processes in space, namely random fields, which are characterized quantitatively by the spatial functions of the first two statistical moments, such as the mean and covariance. As the results of analysis, variabilities of structural responses are also quantified by such statistical moments. A useful comparative review of this method is provided by Benaroya and Rehak,[1] and advanced developments were presented by Liu et al.[2–4] Concerning the development of SFEM for composites, there are only a few papers.[5,6] Although these studies considered the various composite random parameters such as fiber angles, layer thicknesses, and material properties, some problems remain. Random thickness affects strains of each lamina because the strains are functions of thickness directional coordinate. Under the condition of random thickness, the strain-displacement relations should be considered as random equations. Also, random fiber angles of laminae will affect not only stiffness change of a laminate but also the evaluation of strains and stresses in material directions, which are important for failure estimation of composites. These problems have not been pointed out in previous studies.

In reliability analysis for composites, uncertainties of structural parameters must be considered because the inherent uncertainties of composites are quite large and have large effects on the reliability of structures. Most of the research concerning reliability of composites,[7–10] however, considered only applied loads and strengths as random, and uncertainties of a structure itself were not

Received May 13, 1995; revision received Dec. 10, 1995; accepted for publication Feb. 7, 1996.

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considered. Also, they dealt with only the laminates under simple in-plane tension and shear forces using the classical lamination theory. System reliability was not evaluated because the correlations between possible multiple failure modes were not considered.

In this paper, the SFEM for laminated composites is developed considering the aforementioned problems based on the second moment method. Material properties, fiber angles, laminate thickness, and curvatures are considered as random parameters. The curvatures mean initial curvature as a result of manufacturing error for the case of a flat plate. The shell element based on the first-order shear deformation shell theory[11] is employed. In the formulation, the strain-displacement relations are considered as random equations, and also included is the effect of randomness of material coordinates because of random fiber orientation. Using the statistical moments of responses obtained by the SFEM, system reliability of random composite structures is evaluated based on the second moment approach.[12,13] Composite laminates are assumed to be a multicomponent series system, and the bimodal bound of probability of system failure is derived by considering correlations between possible failure modes of different components.

Numerical results are presented for two example problems. The validity and accuracy of the method are verified by comparing the results with those obtained by the Monte Carlo simulation (MCS). The developed SFEM gives accurate statistical results, irrespective of the kind of responses, and even for moderately large variance of random parameters. Narrow and consistent bimodal bounds of the failure probability are obtained in acceptable agreement with the failure probability obtained by the MCS.

Stochastic Finite Element Formulation

Governing Equations

The matrix equation of linear static problem resulting from the finite element discretization is

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F} \tag{1}$$

where

$$\boldsymbol{K} = \int_{\Omega} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} \, \mathrm{d}\Omega$$

All matrices and vectors are functions of random fields $b(\boldsymbol{x})$, i.e., random functions. Especially, the matrix $\boldsymbol{B}$ is treated as a random function of random thickness and curvatures. Each of the composite random parameters is modeled as one correlated random field $b(\boldsymbol{x})$, which is discretized to a set of finite number of random variables b_i and their covariance matrix as follows:

$$b(\boldsymbol{x}) = \sum_{i=1}^{q} N_i(\boldsymbol{x}) b_i \tag{2}$$

$$\mathrm{Cov}(b_i, b_j) = \sqrt{\mathrm{Var}(b_i)\mathrm{Var}(b_j)}\, R(\boldsymbol{x}_i, \boldsymbol{x}_j) \tag{3}$$

where b_i are the values of $b(\boldsymbol{x})$ at some discretization points. The random functions $\boldsymbol{K}, \boldsymbol{u}$, and $\boldsymbol{F}$ in Eq. (1) are expanded about $\bar{b}$ via Taylor series and retain only up to second-order terms:

$$\boldsymbol{K} = \bar{\boldsymbol{K}} + \sum_{i=1}^{q} \bar{\boldsymbol{K}}_{b_i} \Delta b_i + \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{q} \bar{\boldsymbol{K}}_{b_i b_j} \Delta b_i \Delta b_j \tag{4}$$

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \sum_{i=1}^{q} \bar{\boldsymbol{u}}_{b_i} \Delta b_i + \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{q} \bar{\boldsymbol{u}}_{b_i b_j} \Delta b_i \Delta b_j \tag{5}$$

$$\boldsymbol{F} = \bar{\boldsymbol{F}} + \sum_{i=1}^{q} \bar{\boldsymbol{F}}_{b_i} \Delta b_i + \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{q} \bar{\boldsymbol{F}}_{b_i b_j} \Delta b_i \Delta b_j \tag{6}$$

where

$$\Delta b_i = (b_i - \bar{b}_i)$$

where subscript b_i represents the partial derivative with respect to b_i and $b_i b_j$ denotes the mixed partial derivative with respect to b_i and b_j. The overbar implies the evaluation of that term at $\bar{b}$. Substituting Eqs. (4–6) into Eq. (1) and equating equal-order terms of Δb, one can obtain the governing equations as follows:

Zeroth-order equation:

$$\bar{\boldsymbol{K}} \bar{\boldsymbol{u}} = \bar{\boldsymbol{F}} \tag{7}$$

First-order equations:

$$\bar{\boldsymbol{K}} \bar{\boldsymbol{u}}_{bi} = \bar{\boldsymbol{F}}_{1i} \qquad i = 1, 2, \ldots, q \tag{8}$$

where

$$\bar{\boldsymbol{F}}_{1i} = \bar{\boldsymbol{F}}_{b_i} - \bar{\boldsymbol{K}}_{b_i} \bar{\boldsymbol{u}} \tag{9}$$

Second-order equation:

$$\bar{\boldsymbol{K}} \bar{\boldsymbol{u}}_2 = \bar{\boldsymbol{F}}_2 \tag{10}$$

where

$$\bar{\boldsymbol{u}}_2 = \frac{1}{2} \sum_{i,j=1}^{q} \bar{\boldsymbol{u}}_{b_i b_j} \mathrm{Cov}(b_i, b_j) \tag{11}$$

$$\bar{\boldsymbol{F}}_2 = \sum_{i,j=1}^{q} \left\{ \frac{1}{2} \bar{\boldsymbol{F}}_{b_i b_j} - \frac{1}{2} \bar{\boldsymbol{K}}_{b_i b_j} \bar{\boldsymbol{u}} - \bar{\boldsymbol{K}}_{b_i} \bar{\boldsymbol{u}}_{b_j} \right\} \mathrm{Cov}(b_i, b_j) \tag{12}$$

where the second-order equation has been multiplied by expectation operator $E[\cdot]$. Once $\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}_{b_i}$, and $\bar{\boldsymbol{u}}_2$ are obtained by solving Eqs. (7), (8), and (10), respectively, the statistical moments of responses are obtained sequentially.

Statistical Moments of Responses

From Eq. (5), the second-order approximation of mean value and the first-order approximation of covariance of displacements are given by

$$E[\boldsymbol{u}] = \bar{\boldsymbol{u}} + \frac{1}{2} \sum_{i,j=1}^{q} \bar{\boldsymbol{u}}_{b_i b_j} \mathrm{Cov}(b_i, b_j) \tag{13}$$

$$\mathrm{Cov}(\boldsymbol{u}, \boldsymbol{u}) = \sum_{i,j=1}^{q} \bar{\boldsymbol{u}}_{b_i} \bar{\boldsymbol{u}}_{b_j}^T \mathrm{Cov}(b_i, b_j) \tag{14}$$

For strains and stresses, the statistical moments of strains in global coordinates ε are first calculated, and then these are transformed to material coordinates to obtain strains and stresses in material coordinates ε' and σ'. The strain-displacement relation in global coordinates is given by

$$\varepsilon = \boldsymbol{B}\boldsymbol{u} \tag{15}$$

Since the matrix $\boldsymbol{B}$ is a function of thickness coordinate and curvatures in the laminated composite shells, Eq. (15) is a random equation in general. Therefore, ε is also expanded as follows:

$$\varepsilon = \bar{\varepsilon} + \sum_{i=1}^{q} \bar{\varepsilon}_{b_i} \Delta b_i + \frac{1}{2} \sum_{i,j=1}^{q} \bar{\varepsilon}_{b_i b_j} \Delta b_i \Delta b_j \tag{16}$$

where

$$\bar{\varepsilon} = \bar{\boldsymbol{B}} \bar{\boldsymbol{u}} \tag{17a}$$

$$\bar{\varepsilon}_{b_i} = \bar{\boldsymbol{B}}_{b_i} \bar{\boldsymbol{u}} + \bar{\boldsymbol{B}} \bar{\boldsymbol{u}}_{b_i} \tag{17b}$$

$$\bar{\varepsilon}_{b_i b_j} = \bar{\boldsymbol{B}}_{b_i b_j} \bar{\boldsymbol{u}} + 2\bar{\boldsymbol{B}}_{b_i} \bar{\boldsymbol{u}}_{b_j} + \bar{\boldsymbol{B}} \bar{\boldsymbol{u}}_{b_i b_j} \tag{17c}$$

Analogous to the case of displacements, from Eq. (16),

$$E[\varepsilon] = E[\bar{\varepsilon}] + \frac{1}{2} \sum_{i=1}^{q} \sum_{j=1}^{q} \bar{\varepsilon}_{b_i b_j} \mathrm{Cov}(b_i, b_j) \tag{18}$$

$$\mathrm{Cov}(\varepsilon, \varepsilon) = E[(\varepsilon - \bar{\varepsilon})(\varepsilon - \bar{\varepsilon})^T] = \sum_{i=1}^{q} \sum_{j=1}^{q} \bar{\varepsilon}_{b_i} \bar{\varepsilon}_{b_j}^T \mathrm{Cov}(b_i, b_j) \tag{19}$$

Substituting Eqs. (17a–17c) into Eqs. (18) and (19), the first two statistical moments of ε are obtained from those of displacements and additional terms as follows:

$$E[\varepsilon] = \bar{B}E[u] + \sum_{i,j=1}^{q} (\bar{B}_{b_i}\bar{u}_{b_j} + \tfrac{1}{2}\bar{B}_{b_ib_j}\bar{u})\mathrm{Cov}(b_i, b_j) \tag{20}$$

$$\mathrm{Cov}(\varepsilon, \varepsilon) = \bar{B}\mathrm{Cov}(u, u)\bar{B}^T + \sum_{i,j=1}^{q} \Big[(\bar{B}_{b_i}\bar{u})(\bar{B}_{b_j}\bar{u})^T + (\bar{B}_{b_i}\bar{u})(\bar{B}\bar{u}_{b_j})^T + (\bar{B}\bar{u}_{b_j})(\bar{B}_{b_i}\bar{u})^T\Big]\mathrm{Cov}(b_i, b_j) \tag{21}$$

For the composite laminates, strain and stress components in the material direction are important in failure estimation. They are obtained from ε sequentially as follows:

$$\varepsilon' = T\varepsilon \tag{22}$$

$$\sigma' = Q\varepsilon' \tag{23}$$

Since T and Q are the functions of fiber angle and material properties of a lamina, respectively,[14] Eqs. (22) and (23) are random equations when these parameters are random. Therefore they also are expanded similarly to ε. The mean value and covariance of ε' are obtained from those of ε by

$$E[\varepsilon'] = \bar{T}E[\varepsilon] + \sum_{i,j=1}^{q}]\left(\bar{T}_{b_i}\bar{u}_{b_j} + \frac{1}{2}\bar{T}_{b_ib_j}\bar{u}\right)\mathrm{Cov}(b_i, b_j) \tag{24}$$

$$\mathrm{Cov}(\varepsilon', \varepsilon') = \bar{T}\mathrm{Cov}(\varepsilon, \varepsilon)\bar{T}^T + \sum_{i,j=1}^{q} \Big[(\bar{T}_{b_i}\bar{\varepsilon})(\bar{T}_{b_j}\bar{\varepsilon})^T + (\bar{T}_{b_i}\bar{\varepsilon})(\bar{T}\bar{\varepsilon}_{b_j})^T + (\bar{T}\bar{\varepsilon}_{b_i})(\bar{T}_{b_j}\bar{\varepsilon})^T\Big]\mathrm{Cov}(b_i, b_j) \tag{25}$$

Similarly, the statistical moments of σ' can be obtained from statistical moments of ε' by

$$E[\sigma'] = \bar{Q}E[\varepsilon'] + \sum_{i,j=1}^{q} \left(\bar{Q}_{b_i}\bar{\varepsilon}'_{b_j} + \frac{1}{2}\bar{Q}_{b_ib_j}\bar{\varepsilon}'\right)\mathrm{Cov}(b_i, b_j) \tag{26}$$

$$\mathrm{Cov}(\sigma', \sigma') = \bar{Q}\,\mathrm{Cov}(\varepsilon', \varepsilon')\bar{Q}^T + \sum_{i,j=1}^{q} \Big[(\bar{Q}_{b_i}\bar{\varepsilon}')(\bar{Q}_{b_j}\bar{\varepsilon}')^T + (\bar{Q}_{b_i}\bar{\varepsilon}')(\bar{Q}\bar{\varepsilon}'_{b_j})^T + (\bar{Q}\bar{\varepsilon}'_{b_i})(\bar{Q}_{b_j}\bar{\varepsilon}')^T\Big]\mathrm{Cov}(b_i, b_j) \tag{27}$$

where

$$\bar{\varepsilon}' = \bar{T}\bar{\varepsilon} \tag{28a}$$

$$\bar{\varepsilon}'_{b_i} = \bar{T}_{b_i}\bar{\varepsilon} + \bar{T}\bar{\varepsilon}_{b_i} \tag{28b}$$

$$\bar{\varepsilon}'_{b_ib_j} = \bar{T}_{b_ib_j}\bar{\varepsilon} + 2\bar{T}_{b_i}\bar{\varepsilon}_{b_j} + \bar{T}\bar{\varepsilon}_{b_ib_j} \tag{28c}$$

The present formulations can be interpreted as the case with one random field. For multiple random fields, the terms multiplied by $\mathrm{Cov}(b_i, b_j)$ in all equations are determined by superposition of those for individual random fields. For each correlated random field, the covariance matrix $\mathrm{Cov}(b_i, b_j)$ will be a full matrix, and therefore it will require too many computations. To reduce computations, the orthogonal transformation procedure proposed by Liu et al.[2–4] is used. The correlated vector b_i is transformed to an uncorrelated vector through an eigenvalue orthogonalization resulting in a diagonal variance matrix. The mixed derivatives and double summations in i and j appearing in all equations reduce to second derivatives and single summations, respectively.

The present formulations can be applied to any general plate or shell elements. In this study, the layered shell element based on the first-order shear deformation shell theory[11] is employed. This element is efficient since integration in the thickness direction is derived analytically. Also, it is simple to consider the curvatures as random parameters because the element stiffness matrix is given by the functions of laminate curvatures.

Random Parameters of Composite Laminates

In this study, the composite parameters of material properties, thickness of laminate, fiber angles of laminae, and curvatures of laminate are considered as random parameters. The derivatives of global stiffness matrix K with respect to these random parameters $\bar{K}_{b_i}$ and $\bar{K}_{b_ib_j}$ can be evaluated analytically by performing differentiation in proper steps of process of calculating K according to the kind of random parameters. In this section, discussions on the assumptions for composite random parameters and the evaluation of derivatives of K are presented.

Material Properties

For the layered shell element, the number of independent material properties are six, which are elastic moduli E_1, E_2, G_{12}, G_{23}, and G_{31}, and Poisson's ratio ν_{12}. All of these can be considered as random parameters in the same manner. The differentiation of K with respect to these parameters is carried out by differentiating analytically the reduced stiffness matrix Q, which is a stress-strain matrix in material coordinates and the function of only the six material properties.[14]

Fiber Angles of Laminae

The fiber angles of all laminae are treated as random parameters. They are assumed to be statistically independent supposing a situation that all of the laminae are tailored and laminated independently. The differentiation of K with respect to the fiber angle is performed in the process of calculating transformed stiffness matrix Q'_k of kth lamina. The matrix Q'_k is a function of the fiber angle of a lamina and given by transforming Q from lamina material coordinates to global laminate coordinates.[14]

Thickness of Laminate

In the previous studies,[5,6] it was assumed that the thicknesses of all layers are random but the total thickness of a laminate is kept constant. The real phenomena, however, show that the thicknesses of laminates vary considerably with locations even in one laminate. Furthermore, the previous assumption requires too many computations as the number of layers increases. In this study, the total thickness of a laminate is considered as one random parameter. It is assumed that lamina thicknesses also vary according to the variation of laminate thickness, maintaining constant ratio to the total thickness of laminate. This assumption saves computation since the randomness of the thickness is modeled by only one random field. The differentiation of K is carried out analytically in the laminate extensional, coupling, and bending stiffness matrices. The procedure is explained only for the bending stiffness matrix D for brevity. It is defined by

$$D = \frac{1}{3}\sum_{k=1}^{p} Q'_k(z_k^3 - z_{k-1}^3) \tag{29}$$

where p is the number of laminae and z_k is the thickness coordinate from the midsurface of laminate to the upper boundary of the kth lamina. For differentiation with respect to the total thickness of laminate t, it should be rewritten as

$$D = \frac{1}{3}\sum_{k=1}^{p} Q'_k\left[\left(\frac{z_k}{t}\right)^3 - \left(\frac{z_k - 1}{t}\right)^3\right]t^3 \tag{30}$$

where both the z_k and t are random variables, but z_k/t is constant by assuming that the lamina thickness maintains constant ratio to the total thickness. The first-order derivative of D with respect to t is obtained by

$$\frac{\partial D}{\partial t} = \sum_{k=1}^{p} Q'_k\left[\left(\frac{z_k}{t}\right)^3 - \left(\frac{z_k - 1}{t}\right)^3\right]t^2 \tag{31}$$

Curvatures of Laminate

Precise control of curvature is quite difficult, particularly for thin laminates. The uncertainty of curvature arises from various sources during manufacturing, such as inaccuracy of fiber angles, nonuniform thermal conduction and chemical shrinkage, and residual thermal stresses. In this study, the curvatures in global x and y directions of laminates are treated as random parameters. For the case of flat plates, random curvatures are the measure for the degree of flatness. The derivatives of $\boldsymbol{K}$ are evaluated in the process of calculating element stiffness matrices, which are given by the functions of curvatures.[11]

Bimodal Bounds of System Reliability

Using the statistical moments of structural responses obtained by the SFEM, reliability analysis can be made based on the second moment approach.[12,13] In the reliability analysis, strengths are also modeled as random fields. Failure of a laminated composite structure generally involves many different failure modes. Failures in different positions and laminae constitute distinct and different failure modes of the laminate system. Failures of any position are again distinguished into three different failure modes of fiber, transverse, and shear failure according to failure directions in material coordinates. In this regard, composite laminates can be defined as a multicomponent system with multiple failure modes in which the number of components are assumed to be equal to the number of different failure modes. If we assume that all stress components calculated in the finite element model of a laminate can cause individual failure events, the number of different failure modes is defined as follows: number of different failure modes = number of laminae × number of elements × number of Gauss points per element × number of stress components.

The number of failure modes obviously depends on the finite element mesh. However, considering correlations between failure modes of different points, consistent reliability for the entire system can be evaluated. It will be verified in the numerical examples.

For all single failure modes of a laminate, probabilities of failure are evaluated using the first and second statistical moments of stresses obtained by Eqs. (26) and (27). The maximum stress failure criterion is employed as the limit-state function of a single failure mode. For the ith failure mode, it is represented as

$$g_i = X_i - Y_i > 0; \quad \text{safe state} \quad i = 1, 2, \ldots, m$$
$$\leq 0; \quad \text{failure state} \tag{32}$$

where X_i and Y_i are stress and strength corresponding to the ith failure mode, respectively. They are obviously statistically independent. They are assumed to be normal variables and transformed to standard normal. The reliability index for the ith failure mode is then given as the minimum distance from the origin to a limit-state plane in the space of standard normal variables. As the limit-state function is linear, it is obtained analytically as follows:

$$\beta_i = \frac{\mu_{X_i} - \mu_{Y_i}}{\sqrt{\sigma_{X_i}^2 + \sigma_{Y_i}^2}} \tag{33}$$

The failure probability of the ith failure mode is given by $P_{f_i} = \Phi(-\beta_i)$. If the X_i and Y_i are normal, then P_{f_i} is exact. Otherwise P_{f_i} is approximation.

To obtain the bimodal bound of failure probability of a system, correlations between different failure modes have to be evaluated. As stresses and strengths are uncorrelated, they can be obtained by

$$\rho_{ij} = \frac{\mathrm{Cov}(g_i, g_j)}{\sigma_{g_i}\sigma_{g_i}} = \frac{\mathrm{Cov}(X_i, X_j) + \mathrm{Cov}(Y_i, Y_j)}{\sqrt{\sigma_{X_i}^2 + \sigma_{Y_i}^2}\sqrt{\sigma_{X_j}^2 + \sigma_{Y_j}^2}} \tag{34}$$

where $\mathrm{Cov}(X_i, X_j)$ are calculated with Eq. (27) in the stochastic finite element analysis, and $\mathrm{Cov}(Y_i, Y_j)$ are given from appropriately assumed random fields of strengths. With the single mode failure probabilities P_{f_i} and correlations between them, we can evaluate bimodal bounds of the failure probability of a laminate system as follows:

$$P_{f_1} + \max\left[\sum_{i=2}^{m}\left\{P_{f_i} - \sum_{j=1}^{i-1} P(E_i E_j)\right\}; 0\right]$$
$$\leq P_f \leq \sum_{i=1}^{m} P_{f_i} - \sum_{i=2}^{m} \max_{j<i} P(E_i E_j) \tag{35}$$

where P_{f_i} are ordered in the sequence of the larger value of the failure probability and P_{f_1} is the largest one. The joint probability $P(E_i E_j)$ for a joint event of E_i and E_j are again bounded as follows:

$$\max[P(A), P(B)] \leq P(E_i E_j) \leq P(A) + P(B) \tag{36}$$

where

$$P(A) = \Phi(-\beta_i)\Phi\left(-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1 - \rho_{ij}^2}}\right) \tag{37a}$$

$$P(B) = \Phi(-\beta_j)\Phi\left(-\frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1 - \rho_{ij}^2}}\right) \tag{37b}$$

The $P(E_i E_j)$ in Eq. (35) is then approximated with the appropriate sides of Eq. (36); that is, for the lower bound of Eq. (35), $P(A) + P(B)$ is used, whereas $\max[P(A), P(B)]$ is used for the upper bound.

In Eq. (35), m is the number of different failure modes defined earlier. It will become very large for the laminates composed of many laminae or for the finite element mesh involving many elements. The computation of $P(E_i E_j)$ for the pairs of all components will be a formidable task. However, in many cases, all of the failure modes do not contribute to the failure of the system; therefore it is sufficient to involve only the several potential failure modes in practical calculation. The number of dominant failure modes can be determined with relative magnitude of single mode failure probabilities to the largest one; that is $P_{f_m}/P_{f_1} > e$. Using the values of 10^{-2}–10^{-4} for e, enough dominant failure modes are involved to yield accurate results.

The underlying assumption in the preceding procedure is that composite laminates are a multicomponent series system such that the failure of any one or more of the components constitutes the failure of the system. Therefore, system failure probability P_f in Eq. (35) can be interpreted as the probability of initial damage by any one or more failure modes in laminates.

Numerical Examples

The procedures developed in the preceding sections are verified through two example problems by comparing the results with those of MCS of 60,000 realizations. Table 1 shows the selected random parameters to be modeled as random fields and their statistics. For the material properties, only the fiber directional stiffness is treated as a random parameter for computational savings. The values of the other material properties are as follows: $E_2 = 9.4$ GPa, $G_{12} = G_{13} = 4.2$ GPa, $G_{23} = 3.1$ GPa, and $\nu_{12} = 0.28$. In the reliability analysis, the strength components are also considered as random parameters. The mean values of strengths used are as follows: $X_T = 1726$ MPa, $X_C = -1051$ MPa, $Y_T = 61$ MPa, $Y_C = -141$ MPa, and $S = 88$ MPa. The coefficients of variation are assumed to be 0.1 for all of the strength components. The autocorrelation coefficient function of random fields is assumed to be exponentially decaying, which is defined by

$$R(\boldsymbol{x}_i, \boldsymbol{x}_j) = \exp\left[-\left(\frac{x_i - x_j}{\lambda_x}\right)^2 - \left(\frac{y_i - y_j}{\lambda_y}\right)^2\right] \tag{38}$$

As decay factors λ_x and λ_y increase, the correlation becomes strong and vice versa.

Table 1 Statistics of the random parameters of composite laminates

Random parameter	Mean	Standard deviation	Coefficient of variation
E_1(stiffness in fiber direction)	127.8 GPa	6.39 GPa	0.05
t (total thickness)	1.0 mm	0.05 mm	0.05
κ_x and κ_y (curvatures)	0	10^{-5} mm^{-1}	——
Fiber angle of each layer, deg	——	2	——

Simply Supported Laminated Plate Under Uniform Pressure

The problem is described in Fig. 1. Uniformly spaced $n \times n$ meshes of eight-node quadratic elements are used. Random fields are discretized to the random variables at the center of elements so that the number of random variables q is equal to the number of elements. In this example, the spatial correlation of random fields is considered in three different levels, such as perfectly correlated ($\lambda_x = \lambda_y = \infty$), partially correlated ($\lambda_x = \lambda_y = 70$ mm) and uncorrelated ($\lambda_x = \lambda_y = 0$). The curvatures of a laminate are always considered as perfectly correlated random fields in all of the cases. This is consistent with real phenomena.

The standard deviations of selected structural responses are shown in Figs. 2 and 3a–3c. For both the perfectly correlated case and the uncorrelated case, the results of SFEM are compared with those of MCS. The mean values of responses are omitted for brevity, which are much more accurate than the variance because the mean values are second order approximated whereas the variances are first order approximated. The results of SFEM show excellent agreement with those of MCS, irrespective of the kind of responses and the correlation strength of random fields. The accuracy of results for σ', which are calculated lastly as the functions of all of the previously calculated responses, is nearly the same as that for displacement, which is the first obtained response. Therefore, the error in the sequential procedure of first- and second-order approximation to obtain the response statistics is very small. The randomness of thickness, curvatures, and fiber angles has a large effect on the evaluation of the statistics of strains and stresses; therefore accurate results can be obtained only when $\boldsymbol{B}$ and $\boldsymbol{T}$ matrices are considered to be random operators as proposed in this study, or else a considerable error is introduced.

In these results, two limiting cases of correlation strength are of particular importance since these results do not require the knowledge of the autocorrelation of random fields. It is rather difficult to estimate the practical autocorrelation for the random fields such as material properties and fiber angles. In this example, the results for the perfectly correlated case can be interpreted as the upper bound on the variability of responses.

The preceding results are for the 6×6 mesh. In the system reliability analysis using the SFEM, a different finite element mesh size may give a different bound of failure probability because the number of failure modes involved in the bimodal bound estimation depends on the number of finite elements. To examine this problem, the bimodal bounds of P_f were calculated for different meshes, 4×4, 6×6, 8×8, 10×10, and 12×12, and the results are shown in

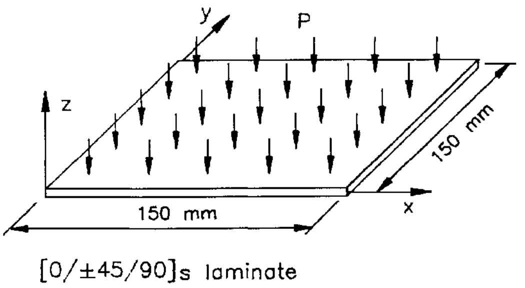


Fig. 1 Simply supported laminated plate under uniform pressure. Simply supported boundary conditions; $u = v = w = 0$ at all boundary edges.

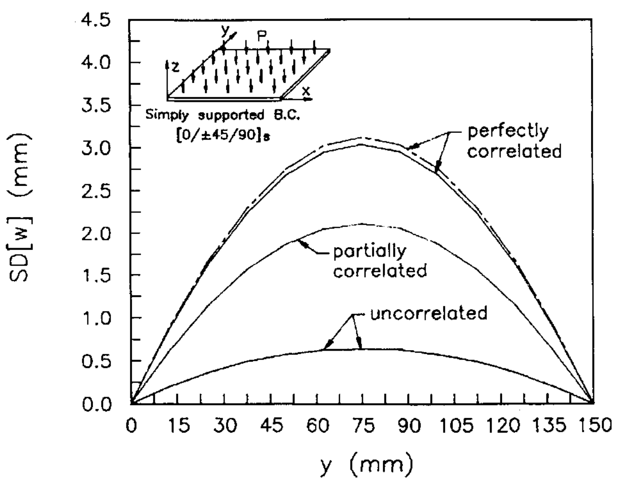
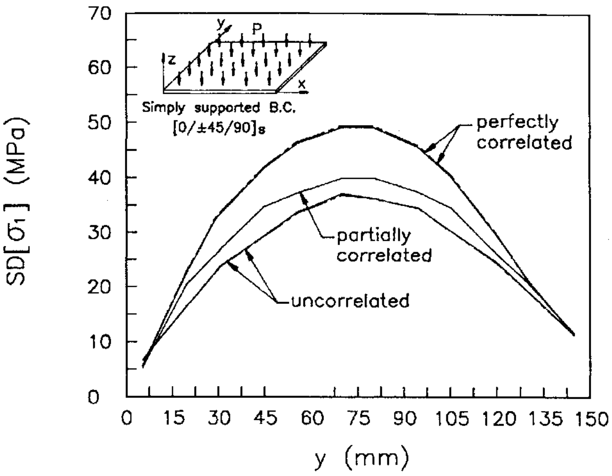
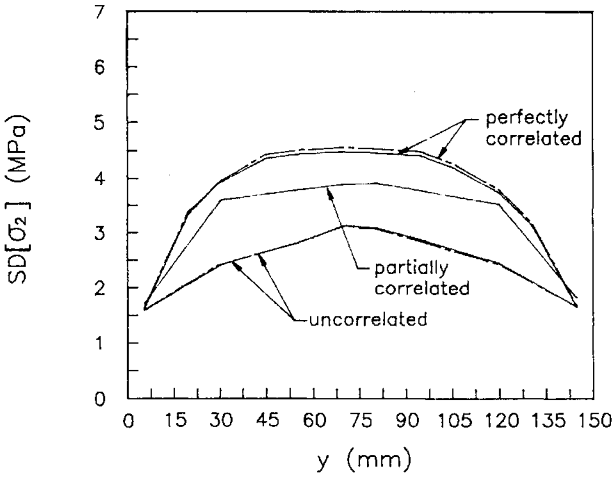


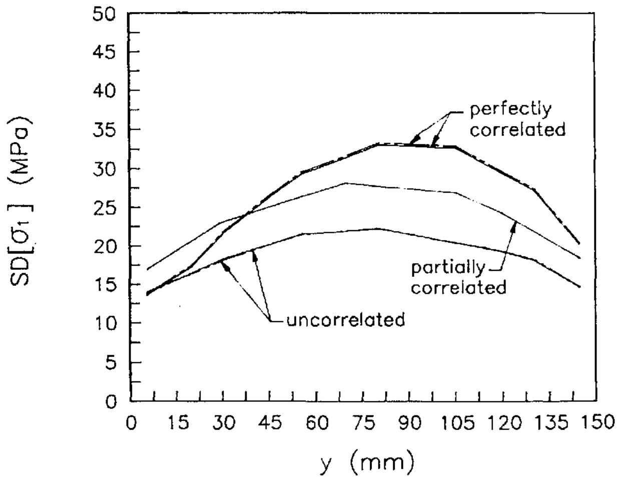
Fig. 2 Standard deviations of displacement w for three different levels of correlation of random fields along y coordinate at $x = 75$ mm, compared with MCS for two limiting cases; applied pressure $P = 40$ kPa: — - —, MCS and ——, SFEM.



a) σ_1 for 0-deg bottom ply



b) σ_2 for 0-deg bottom ply



c) σ_1 for 45-deg second ply

Fig. 3 Standard deviations of stresses in material coordinates for three different levels of correlation of random fields along y coordinate at $x = 69.7$ mm, compared with MCS for two limiting cases; applied pressure $P = 40$ kPa: — - —, MCS and ——, SFEM.

Table 2 Comparison of bimodal bounds of P_f with MCS for simply supported laminated plate under uniform pressure in two loading cases and two limiting cases of random fields

p, kPa		SFEM	MCS
40.0	Perfectly correlated	$0.225 \times 10^{-2} < P_f < 0.338 \times 10^{-2}$	0.392×10^{-2}
(SF = 1.5)	Uncorrelated	$0.704 \times 10^{-2} < P_f < 0.709 \times 10^{-2}$	0.480×10^{-2}
47.9	Perfectly correlated	$0.460 \times 10^{-1} < P_f < 0.695 \times 10^{-1}$	0.516×10^{-1}
(SF = 1.3)	Uncorrelated	$0.190 < P_f < 0.228$	0.146

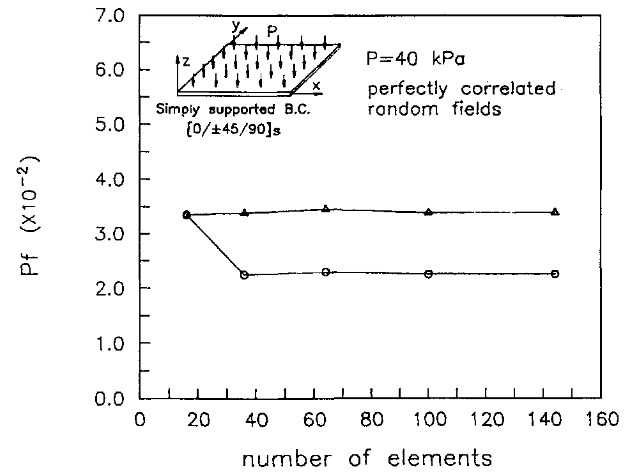


Fig. 4 Independency of probability bounds of system failure on the number of elements: △, upper bound and ○, lower bound.

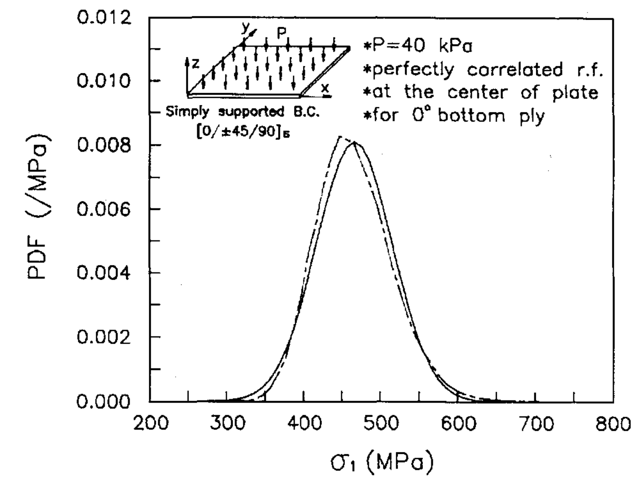


Fig. 6 Probability density functions of fiber directional stress σ_1: — - —, MCS and ——, SFEM.

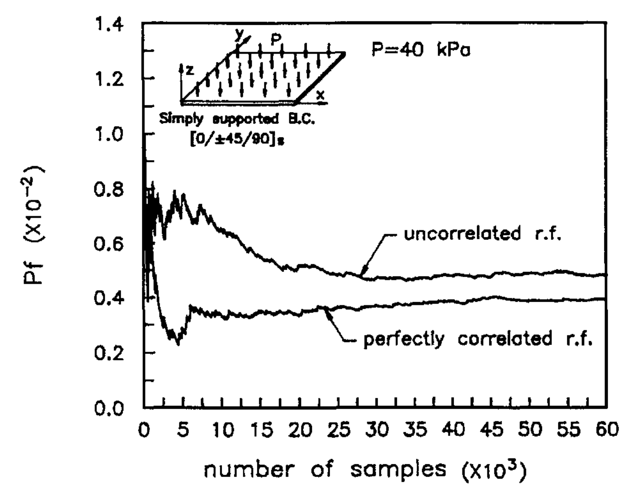


Fig. 5 Convergency of probability of system failure along the number of samples in the MCS.

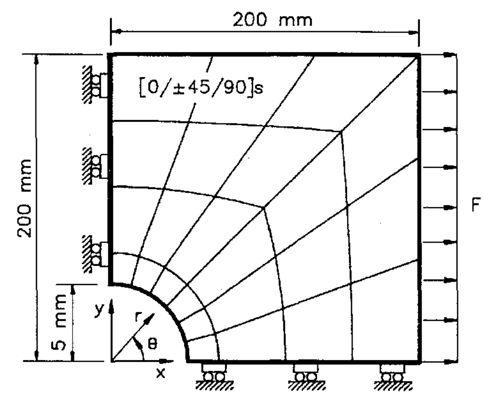


Fig. 7 Laminated composite plate with hole in center subjected to uniform tension.

Fig. 4. The present method gives nearly the same results for different meshes except for the 4 × 4 mesh. The different bound of P_f of the 4 × 4 mesh is a result of the stress fields that do not converge yet in this case. Once the stress fields converge, the number of elements does not affect the evaluation of probability bounds of system failure because of the consideration of the correlations between failure modes in different positions.

The probability bounds of system failure derived by the present method can be validated by comparing them with the failure probability obtained by MCS. The results are compared in Table 2 for the cases of different loadings of $P = 40$ and 47.9 kPa, which correspond to the loads with the safety factors of 1.5 and 1.3 in deterministic regard, respectively. Because the validity of MCS results depends on the number of samples, convergency of the results along the number of samples was checked in Fig. 5. For the case of $P = 47.9$ kPa and a perfectly correlated random field, the probability bound by SFEM is in good agreement with the failure probability from MCS. The results of the other cases are in acceptable agreement. The difference between the results of SFEM and MCS resulted because the reliability analysis based on second moment approach approximates the probability distribution of stresses to normal, whereas the real probability distribution of stresses simulated by MCS is not exactly normal. The normal distribution of stress obtained from the first two moments of SFEM is compared with the probability density function obtained statistically from MCS in Fig. 6. Although some errors in the probability distributions and failure probabilities are inevitable in the second moment method, the proposed method is very efficient and accurate enough to be applied to engineering problems. Even for this simple problem, MCS requires tremendous computations, but the proposed method takes only several multiples of the computational time needed in the deterministic finite element analysis.

Laminated Plate with Hole Subjected to Uniform Tension

The square laminate with a hole in the center is considered. The problem is described in Fig. 7. The eight-node quadratic elements are used. In real applications, the stochastic finite element analysis and the system reliability analysis require full modeling of the laminate. However, for the full modeling in this example, MCS will require too many computations to obtain reasonable failure probability with enough number of samples. Therefore, only one quadrant

Table 3 Comparison of bimodal bounds of P_f with MCS for laminated composite plate with hole

Case	SFEM	MCS
1 (F = 1565 N, SF = 1.5)	$0.140 \times 10^{-2} < P_f < 0.219 \times 10^{-2}$	0.192×10^{-2}
2 (F = 1174 N, SF = 2.0)	$0.148 \times 10^{-2} < P_f < 0.230 \times 10^{-2}$	0.253×10^{-2}

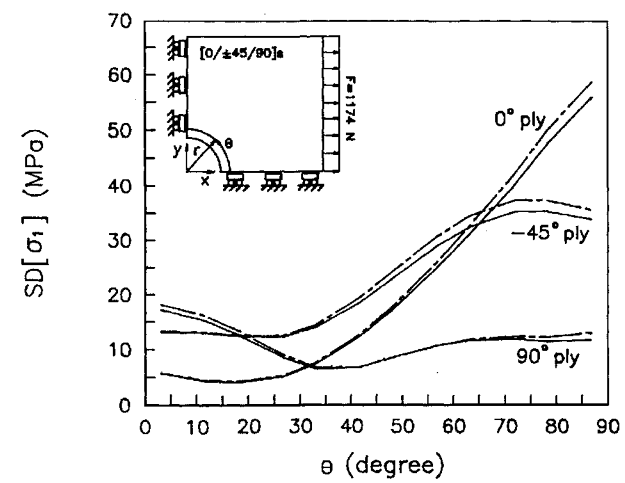


Fig. 8 Standard deviations of fiber directional stresses σ_1 around the hole along θ at r = 5.42 mm: — - —, MCS and ——, SFEM.

of the laminate was analyzed for the comparison with MCS. In this example, the dispersion of random parameters is considered in two cases. In case 1, it is the same as in the previous example, shown in the Table 1. In case 2, the dispersion is increased so that coefficients of variation of E_1 and t are 1.0 and standard deviations of fiber angles are 3 deg. The curvatures are treated as deterministic for both cases since they have no effect in this loading condition. The coefficients of variation of strengths are 1.0 and 1.5 for cases 1 and 2, respectively. As an upper limit case, perfectly correlated random fields are assumed.

For case 2 of relatively large variance, the standard deviations of σ_1 around the hole are compared with those of MCS in Fig. 8, and they are in good agreement. The proposed SFEM gives good results, although the variance of random parameters is moderately large. In Table 3, the bimodal bounds of P_f for both the cases are also compared with failure probabilities by MCS of 60,000 realizations, and they are in good agreement, irrespective of the variance magnitude of random parameters. Because of relatively larger variance of structures, the reliability of case 2 is not improved compared with case 1 in spite of the larger safety factor. The deterministic analysis and safety factor design cannot account for such variance of structures and its effect on the reliability quantitatively. On the other hand, the reliability analysis for random structures assesses the risk quantitatively by considering the variance of structural parameters.

Conclusions

The SFEM for laminated composite structures was formulated based on the second moment method. The various composite parameters were treated as random parameters. Statistical moments of responses were derived in good agreement with those of the MCS by considering the strain-displacement relations as random equation and considering the effect of randomness of material directions. A procedure of system reliability analysis was also presented based on the second moment approach. The narrow and consistent bimodal bounds of probability of system failure were derived by considering the correlations between possible failure modes of different components. The developed methods are efficient and accurate enough to be applied to the engineering problems and can assess the risk of composite structures quantitatively and logically on the basis of the statistical data of structural parameters.

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